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COMMENT

Slowing-down processes in random systems

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Abstract. We make a conjecture for the temperatures at which the slowing-down processes occur in random systems. The estimated values in simple models are in good agreement with the existing simulations.

There is active interest in the field of the dynamics of random systems (Harris and Stinchcombe 1986, Rammal and Benoit 1985, van Hemmen and Morgenstern 1987, Ngai 1980). One of the common features of the so-far investigated systems is the non-exponential relaxation process (or extremely long relaxation time). This phenomenon is characteristic mainly for the low-temperature phase and is often referred to as slowing down or freezing. The temperature at which this slowing down occurs is more or less well defined and characteristic for the investigated system. In this comment we give a conjecture for this freezing temperature and try to explain its origin.

For characterising a random variable it is generally not enough to treat the average value only. This may be satisfactory to obtain the concrete value of a self-averaging quantity but not to describe, e.g., the analytical properties. (For example, the so-called Griffiths (1969) singularity in the diluted Ising ferromagnet is thought to be due to a subsystem which occurs with exponentially low probability (Schwartz 1978).)

It means therefore that we have to take into consideration the whole distribution. It can be done by treating, e.g., the several moments.

We investigate in detail the random-bond Ising model on hypercubic lattices with the following distribution function for the bonds:

$$\rho(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J'). \tag{1}$$

Here $J' = 0$ or $J' = J$ and we refer to the diluted ferromagnet and the $\pm J$ spin glass, respectively.

The quantity which we treat is the magnetisation (and of course its moments) since it serves as a type of order parameter and its behaviour may be characteristic for the whole system. In the $\pm J$ spin-glass case the second moment is thought to serve as order parameter near $p = \frac{1}{2}$ and our consideration involves it too.

Let us define the moments in the following way:

$$A_k = \lim_{N\to\infty} A_{kN} = \lim_{N\to\infty} \overline{\langle S_i \rangle^k}. \tag{2}$$

Here $\overline{}$ and $\langle\ \rangle$ denote the configurational and thermodynamic averages, respectively, and N is the number of spins. The boundary conditions are thought to be involved in the limiting process and they do not alter the following argument. After averaging,

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the system is homogeneous and the sites i do not play any special role and we can express A_{kN} in a more detailed form:

$$A_{kN} = \frac{1}{N}\sum_i \overline{\langle S_i\rangle^k} = \frac{1}{N}\sum_i \overline{\left[Z^{-k}\,\mathrm{Tr}\, S_i^1 S_i^2 \ldots S_i^k \exp\left(\beta \sum_{\langle mn\rangle} J_{mn} \sum_{\alpha=1}^{k} S_m^\alpha S_n^\alpha\right)\right]}$$
$$= \overline{B_{kN}/Z^k} \tag{3}$$

where $\langle mn\rangle$ denotes the nearest-neighbour (or generally the interacting) pairs, Z is the partition function and the definition of B_{kN} is clear from (3). Z^{-k} can be expanded into a power series of $\ln Z$ and we can express the moments as

$$A_k = \lim_{N\to\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\overline{B_{kN}(k \ln Z)^j}. \tag{4}$$

The next step could be the interchange of the summation and the limiting process but it is generally not allowed. In the following we investigate this question in some detail.

First of all we have to realise that the B_{kN} values are independent of the summation index j (the averaging and the summation are always interchangeable). When N is large enough then the $(\ln Z)^j \approx N^j$ asymptotic form can be used since the free energies are extensive. So we are allowed to interchange the two processes if $B_{kN}N^j \to 0$ as $N\to\infty$ for all the possible j values, i.e. if B_{kN} goes to zero faster than any power of N. For high enough temperatures the magnetisation is zero (above the highest possible critical temperature of the different bond relisations) and we get zero for the B_{kN} values too. Lowering the temperature we reach a region where some of the bond realisations already have non-zero magnetisation but their probability is rather low (e.g., because the average bond strength of the realisation differs from the average of the distribution). These low probabilities always go as $\exp(-aN)$ where $a>0$. At the same time the B_{kN} values are exponentially large, $\exp(bN)$, since B_{kN} has the form $B_{kN}\approx\langle s\rangle Z$ and Z is exponentially large. As far as $(b-a)<0$ we are allowed to interchange the two processes. Now we have to realise that exactly the same exponent $(b-a)$ occurs in the average of B_{kN} (i.e. in $\overline{B_{kN}}$) and we check the sign of $(b-a)$ through this average. Let us introduce a new annealed system which has the partition function Z^* (which we will call the effective system):

$$Z^* = \overline{\mathrm{Tr}\exp\left(\beta \sum_{\langle mn\rangle} J_{mn} \sum_{\alpha=1}^{k} S_m^\alpha S_n^\alpha\right)}. \tag{5}$$

The critical temperature of this effective system T_k determines whether $\overline{B_{kN}}$ is zero. (The calculation of the critical temperatures can be done in the same manner as by Kasai and Syozi (1973).) If $T(=1/\beta) > T_k$ then $\overline{B_{kN}}=0$ whereas for lower temperatures $\overline{B_{kN}}$ is non-zero since above T_k we have to average over the whole phase space. In other words it means that $(b-a)$ is negative above this critical temperature of the effective system.

As a conclusion we can say that as far as $T>T_k$ we are allowed to interchange the summation and the thermodynamic limit in (4) and we can even calculate the A_k values which are equal to zero in this case. However, it does not mean that below T_k the $A_k - s$ are non-zero. It means only that the series ceases to be absolutely convergent and we may find rather large fluctuations during the limiting process.

Naturally the question arises whether these fluctuations, which appear in an artificially constructed series expansion, can be seen in a real system too. We think that the answer is affirmative and we argue in the following that the fluctuations of the series appear as slowing down in the dynamics of the system.

To see this connection we have to keep in mind the physical basis of these fluctuations. We have found temperatures below which some bond realisations have a non-zero 'order parameter', i.e. there are frozen spins in the systems and as we increase the probability of their appearance we reach a value where they cause the above fluctuations at the given temperature. In a computer simulation we have a concrete bond realisation which, with high probability, does not belong to the above 'critical' realisations but we can see the slow relaxation in the system.

To resolve this contradiction we have to realise that some parts of the system may behave as the above special realisations although the whole system does not. This means that some spins behave as a 'superspin' (as a whole), i.e. there is a local order in these parts of the system and they relax in a different way to the other parts. The relaxation of the whole system will be influenced by these parts according to the proper probability weights, i.e. according to the size of them. To obtain realistic weights we have to regard a large system or must average over several samples. We would like to emphasise that the slowing down should appear mostly if the systems are not too small.

Another question is whether we have such well defined 'critical' temperatures or we can see a continuous growing of the relaxation times. At these temperatures a new type of relaxation should appear which can be seen, e.g., in the spectrum of the relaxation times. In the early simulation (see, e.g., Kirkpatrick 1977, 1980) these temperatures were identified as critical temperatures.

On the other hand these fluctuations may have some connection with the occurring singularities as we shall now discuss in some detail.

Z^* is defined in such a way that the corresponding system contains no frustration, i.e., we are allowed to use the Griffiths inequality (Griffiths 1972) and we arrive at a series of the $T_k - s$ monotonically increasing with increasing k. On the other hand $T_k < T_c$ for all $k - s$ (where T_c is the critical temperature of the system where $J' = J$) which means that T_c is an accumulation point of the $T_k - s$. This is exactly the same temperature at which the Griffiths singularity occurs (Griffiths 1969, Schwartz 1978). Although it was found in the field dependence we guess that the accumulation of these critical points relates to the same physical phenomenon.

This conjecture can be more or less checked in computer simulations. Comparing the T_k values to the slowing-down temperatures of simulations we have to choose the appropriate k, since the largest fluctuations occur at this temperature, e.g., in the fifth moment at T_5.

In diluted ferromagnets the magnetisation is the most widely investigated quantity (at least in the relaxation studies) which means that we have to compare the observed slowing-down temperatures to T_1. The several T_1 values are depicted as a function of the concentration in figure 1 for the square lattice. We can see that $T_1 > 0$ for all non-zero $p - s$ even for $p < p_c$ (where p_c is the threshold value of the percolation problem at the given lattice; on the square lattice $p_c = \frac{1}{2}$). Such a slowing-down process below p_c was already suggested by Henley (1985) and Rammal and Benoit (1985) and its existence was checked in the site-diluted case by Chowdhury and Stauffer (1986). The concrete value of the slowing-down temperature was estimated by Jain (1986) only. He investigated the bond-diluted case at the threshold concentration and found $T_1 \approx 0.4 T_c$ which is in good agreement with our $0.5 T_c$ value since the former one corresponds rather to a lower bound than to an exact value. (Since the increase of the relaxation time begins very slowly it is not an easy task to observe the exact value of the temperature and we can give only a lower bound from the simulation data.)

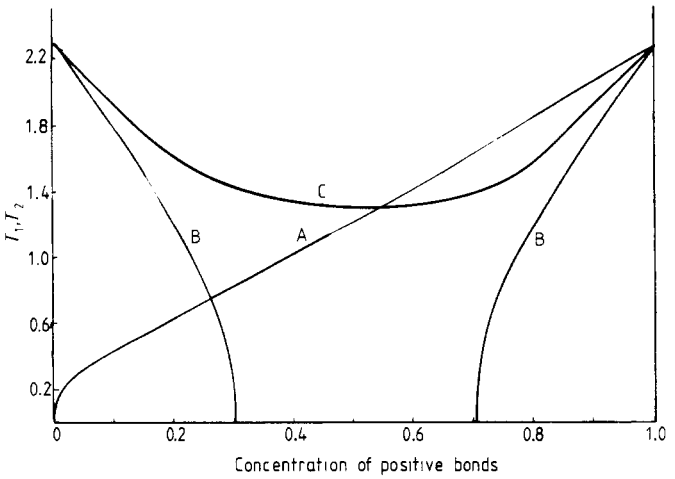

Figure 1. T_1 and T_2 plotted as a function of the concentration of positive bonds, indicating T_1 for a 2D bond-diluted Ising model on the square lattice (curve A) and T_1 (curve B) and T_2 (curve C) both for a 2D $\pm J$ Ising spin glass on the square lattice.

In the $\pm J$ spin glass we have different relevant T_k values depending on the concrete concentration. In figure 1 we see T_1 and T_2 as a function of the concentration of positive bonds. The simulations were made mainly at $p = \frac{1}{2}$ and we may compare the concrete values at this p only. We obtain $T_2 = 1.32J$ in two dimensions and $T_2 = 2.1J$ in three dimensions. These values are in good agreement with the temperatures where the relaxation becomes very slow (Kirkpatrick 1977, 1980, Ogielski 1985).

As a conclusion we made a conjecture for the temperature of the appearance of slowing-down processes and found good agreement with the existing computer simulation data in Ising models. These temperatures could be identified as the critical temperatures of annealed-like effective systems. We argued that the appearance of some 'critical' subsystems are responsible for the slowing-down processes. We believe that similar arguments hold for a wider class of disordered systems.

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